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IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 14985–15000 doi:10.1088/1751-8113/40/50/005

Basic hypergeometric functions and covariant spaces for even-dimensional representations of $U_q[osp(1/2)]$

N Aizawa[1], **R Chakrabarti**[2], **S S Naina Mohammed**[2] **and J Segar**[3]

[1] Department of Mathematics and Information Sciences, Graduate School of Science, Osaka Prefecture University, Nakamozu Campus, Sakai, Osaka 599-8531, Japan
[2] Department of Theoretical Physics, University of Madras, Guindy Campus, Chennai 600 025, India
[3] Department of Physics, Ramakrishna Mission Vivekananda College, Mylapore, Chennai 600 004, India

Received 10 September 2007, in final form 3 November 2007
Published 28 November 2007
Online at stacks.iop.org/JPhysA/40/14985

Abstract
Representations of the quantum superalgebra $U_q[osp(1/2)]$ and their relations to the basic hypergeometric functions are investigated. We first establish Clebsch–Gordan decomposition for the superalgebra $U_q[osp(1/2)]$ in which the representations having no classical counterparts are incorporated. Formulae for these Clebsch–Gordan coefficients are derived, and is observed that they may be expressed in terms of the Q-Hahn polynomials. We next investigate representations of the quantum supergroup $OSp_q(1/2)$ which are not well defined in the classical limit. Employing the universal $\mathcal{T}$-matrix, the representation matrices are obtained explicitly, and found to be related to the little Q-Jacobi polynomials. Characteristically, the relation $Q = -q$ is satisfied in all cases. Using the Clebsch–Gordan coefficients derived here, we construct new noncommutative spaces that are covariant under the coaction of the even-dimensional representations of the quantum supergroup $OSp_q(1/2)$.

PACS numbers: 02.30.Gp, 02.20.Uw

1. Introduction

Soon after the introduction of the quantum groups, their relation to the basic hypergeometric functions via the representation theory was revealed by many authors [1]. In particular, it was observed [2–4] that the representation matrices of the quantum group $SU_q(2)$ can be expressed in terms of the little q-Jacobi polynomials. Kirillov and Reshetikhin demonstrated [5] that the Clebsch–Gordan coefficients of the quantum algebra $U_q[sl(2)]$ relate to the q-Hahn and the dual q-Hahn polynomials. These works furnished a new algebraic framework to the theory of basic hypergeometric functions. Since then, extensive studies on interrelations between the quantum group representations and the basic hypergeometric functions had taken place. We

here mention some key examples of these developments. The matrix elements of the quantum group $SU_q(1,1)$ were found to be related [6] to the polynomials obtained from ${}_2\phi_1$. The realizations of the quantum algebra $U_q[su(1,1)]$ and the generating functions of Al-Salam–Chihara polynomials were found [7] to be linked. The kinship between the group theoretical treatment of the q-oscillator algebra and the q-Laguerre as well as the q-Hermite polynomials was observed [8]. The connection between the metaplectic representation of $U_q[su(1,1)]$ and the q-Gegenbauer polynomials was noted [9].

On the other hand, the study of relations between the quantum supergroups and the basic hypergeometric functions started very recently. In [10], the homogeneous superspaces for the general linear supergroup and the spherical functions on them were investigated. Zou studied [11] the spherical functions on the symmetric spaces arising from the quantum superalgebra $U_q[osp(1/2)]$. This author also observed [12] the relationship between the transformation groups of the quantum super 2-sphere and the little Q-Jacobi polynomials. Considering a 2×2 quantum supermatrix and identifying its dual algebra with the quantum superalgebra $U_q[osp(1/2)]$, the finite-dimensional representations of the quantum supergroup $OSp_q(1/2)$ were found [12] to be related to the little Q-Jacobi polynomials with the assignment $Q = -q$. Instead of Q, the parameter $t = \mathrm{i}\sqrt{q}$ was used in [12]. Adopting an alternate procedure by explicitly evaluating the universal $\mathcal{T}$-matrix that capped the Hopf dual structure, and using the representations of the $U_q[osp(1/2)]$ algebra, the present authors obtained [13] the same result independently. The results in [13] are, however, partial in the sense that only odd-dimensional representations of the algebra $U_q[osp(1/2)]$ are taken into account. One of the purposes of the present study is to incorporate the even-dimensional representations of the supergroup $OSp_q(1/2)$ in the framework of [13]. Continuing our study of the finite-dimensional representations of the universal $\mathcal{T}$-matrix, we observe that the even-dimensional representations of the quantum supergroup $OSp_q(1/2)$ may also be expressed via the little Q-Jacobi polynomials with $Q = -q$. Furthermore, we also study irreducible decomposition of the tensor product of both the even- and odd-dimensional representations of the $U_q[osp(1/2)]$ algebra. Evaluating the Clebsch–Gordan coupling of two even-dimensional representations as well as that of an even- and odd-dimensional representations, we observe that the decomposition is multiplicity free. Proceeding further, we note that the Clebsch–Gordan coefficients for the decompositions are related to the Q-Hahn polynomials with $Q = -q$.

Emergence of the $Q = -q$ polynomials, in contrast to the $Q = q$ polynomials being present for the aforementioned quantum groups, appears to be a generic property of the quantum superalgebra $U_q[osp(1/2)]$. For odd-dimensional representations, this property may be interpreted as a reflection of the isomorphism of $U_q[osp(1/2n)]$ and $U_{-q}[so(2n+1)]$ which holds on the representation space [14]. The even-dimensional representations for which the said isomorphism is not known, however, are still characterized by polynomials with $Q = -q$. Pointing towards a generalized feature of the quantum *supergroups*, the present work puts forward new entries in the list of relations between supergroups and basic hypergeometric functions.

Explicit evaluation of the Clebsch–Gordan coefficients allows us to explore *new noncommutative spaces covariant under the action of even-dimensional representations of* $OSp_q(1/2)$. Employing the method developed in [15], we, for instance, introduce the defining relations of the covariant noncommutative space of dimension four. Our construction may be generalized to describe similar covariant noncommutative spaces of higher dimensions. Especially for the root of unity values of q the representation of these spaces may be of interest in some physical problem.

Our focus on $OSp_q(1/2)$ is explained by physical and mathematical reasons. Physically, fully developed representation theory of the said supergroup may provide better insight to the

solvable vertex-type models [16] endowed with the quantum $U_q[osp(1/2)]$ symmetry, Gaudin models [17] and two-dimensional field theories [18]. Mathematical motivation lies in the fact that $osp(1/2)$ is the simplest superalgebra and a basic building block for other superalgebras.

We plan the paper as follows. In the following section, the definitions and representations of $U_q[osp(1/2)]$ to be used in the subsequent sections are listed. We prove that the even-dimensional representations are of grade star type. The tensor product of two irreducible representations is considered in section 3. We show that the tensor product is decomposed into a direct sum of irreducible representations without multiplicity. Formulae of the Clebsch–Gordan coefficients are derived and is shown that they are expressed in terms of the Q-Hahn polynomials with $Q = -q$. In section 4, the even-dimensional representations of the quantum supergroup $OSp_q(1/2)$ are constructed using the universal $\mathcal{T}$-matrix. Relations of the matrix elements and the little Q-Jacobi polynomials with $Q = -q$ is established. We discuss covariant noncommutative superspaces of dimensions two and four in section 5. Our concluding remarks are given in section 6.

2. $U_q[osp(1/2)]$ and its representations

The quantum superalgebra $\mathcal{U} \equiv U_q[osp(1/2)]$ has been introduced in [19]. The finite-dimensional representations of $\mathcal{U}$, which are q-analogue of the representations of the classical superalgebra $osp(1/2)$, have been investigated in [20, 16]. Classification of the finite-dimensional integrable representations of more general quantum superalgebra $U_q[osp(1/2n)]$ has been made by Zou [21], and therein it has been observed that $U_q[osp(1/2)]$ admits representations which are not deformation of the ones for $osp(1/2)$. Thus, we have two types of finite-dimensional representations for $\mathcal{U}$: one of them has classical counterparts, and the other does not. For the purpose of fixing our notations and conventions, we here list the relations that will be used subsequently.

The algebra $\mathcal{U}$ is generated by three elements H (parity even) and $V_\pm$ (parity odd) subject to the relations

$$[H, V_\pm] = \pm\frac{1}{2}V_\pm, \qquad \{V_+, V_-\} = -\frac{q^{2H} - q^{-2H}}{q - q^{-1}} \equiv -[2H]_q. \tag{2.1}$$

The deformation parameter q is assumed to be generic throughout this article. The Hopf algebra structures defined via the coproduct (Δ), the counit (ϵ) and the antipode (S) maps read as follows:

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \qquad \Delta(V_\pm) = V_\pm \otimes q^{-H} + q^H \otimes V_\pm, \tag{2.2}$$

$$\epsilon(H) = \epsilon(V_\pm) = 0, \tag{2.3}$$

$$S(H) = -H, \qquad S(V_\pm) = -q^{\mp 1/2}V_\pm. \tag{2.4}$$

The finite-dimensional irreducible representations of $\mathcal{U}$ are specified by the highest weight ℓ which takes any non-negative integral or half-integral value. We denote the irreducible representation space of highest weight ℓ by $V^{(\ell)}$. According to [21], a representation is referred to be integrable if $V^{(\ell)}$ is a direct sum of its weight spaces, and if $V_\pm$ act as locally nilpotent operators on $V^{(\ell)}$. The results in [21] for the case of $\mathcal{U}$ can be stated quite simply by introducing an element $K = q^{2H}$ as follows: let $\boldsymbol{v} \in V^{(\ell)}$ be a highest weight vector ($V_+\boldsymbol{v} = 0$). The highest weight representation constructed on $\boldsymbol{v}$ is integrable if and only if

$$K\boldsymbol{v} = \begin{cases} \pm q^\ell & \text{if} \quad \ell \text{ is an integer,} \\ \pm \mathrm{i} q^\ell & \text{if} \quad \ell \text{ is a half-integer,} \end{cases} \tag{2.5}$$

and the integrable representations are completely reducible. The representation in $V^{(\ell)}$ has dimension $2\ell+1$ so that $V^{(\ell)}$ is odd-(even-)dimensional if ℓ is an integer (half-integer). It is known that the classical superalgebra $osp(1/2)$ does not have even-dimensional irreducible representations.

We denote a basis set of $V^{(\ell)}$ as $\left\{e^{\ell}_{m}(\lambda)\,\middle|\, m=\ell,\ell-1,\ldots,-\ell\right\}$, where the index $\lambda=0,1$ specifies the parity of the highest weight vector $e^{\ell}_{\ell}(\lambda)$. The parity of the vector $e^{\ell}_{m}(\lambda)$ equals $\ell-m+\lambda$, as it is obtained by the action of $V_{-}^{\ell-m}$ on $e^{\ell}_{\ell}(\lambda)$. For the superalgebras the norm of the representation basis need not be chosen positive definite. In this work, however, we assume the positive definiteness of the basis elements:

$$\left(e^{\ell}_{m}(\lambda), e^{\ell'}_{m'}(\lambda)\right)=\delta_{\ell\ell'}\delta_{mm'}. \tag{2.6}$$

With these settings, the irreducible representation of $\mathcal{U}$ on $V^{(\ell)}$ is given as follows: for ℓ integer, we take the following form which is a variant of the convention used in [20, 16]:

$$\begin{aligned}
&He^{\ell}_{m}(\lambda)=\frac{m}{2}e^{\ell}_{m}(\lambda),\\
&V_{+}e^{\ell}_{m}(\lambda)=\left(\frac{1}{\{2\}_q}\{\ell-m\}_q\{\ell+m+1\}_q\right)^{1/2}e^{\ell}_{m+1}(\lambda),\\
&V_{-}e^{\ell}_{m}(\lambda)=(-1)^{\ell-m-1}\left(\frac{1}{\{2\}_q}\{\ell+m\}_q\{\ell-m+1\}_q\right)^{1/2}e^{\ell}_{m-1}(\lambda),
\end{aligned} \tag{2.7}$$

where

$$\{m\}_q=\frac{q^{-m/2}-(-1)^m q^{m/2}}{q^{-1/2}+q^{1/2}}. \tag{2.8}$$

The representation space $V^{(\ell)}$ is odd-dimensional. It is known that (2.7) is a grade star representation [22] if $q\in\mathbb{R}$ [16]. The grade adjoint operation is given by

$$H^{*}=H,\qquad V_{\pm}^{*}=\pm(-1)^{\epsilon}V_{\mp}, \tag{2.9}$$

where $\epsilon=\lambda+1 \pmod 2$. The grade adjoint operation is assumed to be an algebra anti-isomorphism and a coalgebra isomorphism.

For a half-integer ℓ, we chose a representation parallel to (2.7) but different from the one in [21]:

$$\begin{aligned}
&He^{\ell}_{m}(\lambda)=\frac{1}{2}(m\pm\eta)e^{\ell}_{m}(\lambda),\\
&V_{+}e^{\ell}_{m}(\lambda)=\pm\left(\frac{1}{\{2\}_q}\{\ell-m\}_q\{\ell+m+1\}_q\right)^{1/2}e^{\ell}_{m+1}(\lambda),\\
&V_{-}e^{\ell}_{m}(\lambda)=(-1)^{\ell-m}i\left(\frac{1}{\{2\}_q}\{\ell+m\}_q\{\ell-m+1\}_q\right)^{1/2}e^{\ell}_{m-1}(\lambda),
\end{aligned} \tag{2.10}$$

where

$$\eta=\frac{\pi\mathrm{i}}{2\ln q}.$$

The factor i appearing in the action of K (2.5) is converted into the constant η for the action of H. We keep two different phase conventions for later convenience. Representation spaces corresponding to each phase choice are denoted by $V_{\pm}^{(\ell)}$. The space $V_{\pm}^{(\ell)}$ is even-dimensional. We come to state our first result. If $q\in\mathbb{R}$, then (2.10) is a grade star representation under the grade adjoint operation

$$H^{*}=H\mp\frac{\pi\mathrm{i}}{2\ln q},\qquad V_{+}^{*}=\pm i(-1)^{\epsilon}V_{-},\qquad V_{-}^{*}=\mp i(-1)^{\epsilon}V_{+}, \tag{2.11}$$

where $\epsilon=\lambda+1 \pmod 2$.

Proof of this statement is rather straightforward. Recall the definition of grade star representation. Let ρ be a representation of a quantum superalgebra U. Denoting a grade star operation by $*$ that is defined in U, the representation ρ is referred to be of the grade star type if $\rho(X)\forall X \in U$ satisfies

$$\rho(X^*) = \rho(X)^*, \tag{2.12}$$

where $\rho(X)^*$ is the superhermitian conjugate defined by

$$\rho(X)^*_{ij} = (-1)^{(\hat{i}+\hat{X})(\hat{i}+\hat{j})}\overline{\rho(X)}_{ji},$$

where $\hat{a}$ denotes the parity of the object a. It is easily verified that (2.10) and (2.11) satisfy (2.12).

3. Clebsch–Gordan decomposition and Q-Hahn polynomials

In this section, we consider the tensor product of two irreducible representations of $\mathcal{U}$. As noted earlier, the algebra $\mathcal{U}$ has two types of grade star representations, namely, odd- and even-dimensional ones. The former maintains one-to-one correspondence to the representation of $osp(1/2)$ of same dimensionality, whereas the latter has no classical analogue. We may consider three cases of the tensor product, namely, the product of two odd-dimensional representations, two even ones, and an odd and an even ones. We observe that for all the three cases the tensor product of two irreducible representations is, in general, reducible, and may be decomposed into a direct sum of irreducible ones without multiplicity:

$$V^{(\ell_1)} \otimes V^{(\ell_2)} = V^{(\ell_1+\ell_2)} \oplus V^{(\ell_1+\ell_2-1)} \oplus \cdots \oplus V^{(|\ell_1-\ell_2|)}. \tag{3.1}$$

The decomposition of the tensored vector space in the irreducible basis is provided by the Clebsch–Gordan coefficients (CGC):

$$e^{\ell}_{m}(\ell_1, \ell_2, \Lambda) = \sum_{\substack{m_1,m_2\\ m_1+m_2=m}} C^{\ell_1\ell_2\ell}_{m_1m_2m} e^{\ell_1}_{m_1}(\lambda) \otimes e^{\ell_2}_{m_2}(\lambda), \tag{3.2}$$

where $\Lambda = \ell_1 + \ell_2 - \ell (\mathrm{mod}\, 2)$ signifies the parity of the highest weight vector $e^{\ell}_{\ell}(\ell_1, \ell_2, \Lambda)$. The decomposition (3.1) is established below by explicit construction of the CGC. Another pertinent problem of interest is the interrelation of the CGC and the basic hypergeometric functions. We prove below that the CGC have polynomial structure corresponding to the Q-Hahn polynomials. We treat three cases separately.

3.1. Two odd-dimensional representations

The Clebsch–Gordon decomposition for $\mathcal{U}$ in this case has been extensively studied in [23]. We here discuss a relation of the CGC and the Q-Hahn polynomials.

Following [24], the basic hypergeometric function ${}_{r+1}\phi_r$ is defined as

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}\middle| Q; z\right] = \sum_{k=0}^{\infty} \frac{(a_1; Q)_k (a_2; Q)_k \cdots (a_{r+1}; Q)_k}{(b_1; Q)_k (b_2; Q)_k \cdots (b_r; Q)_k} \frac{z^k}{(Q; Q)_k}, \tag{3.3}$$

where the shifted factorial reads

$$(x; Q)_k = \begin{cases} 1 & \text{for} \quad k = 0, \\ \prod_{j=0}^{k-1}(1 - xQ^j) & \text{for} \quad k \neq 0. \end{cases} \tag{3.4}$$

The Q-Hahn polynomials are defined [24] via ${}_3\phi_2$ in a standard way:

$$\mathcal{Q}_M(x;a,b,N;Q) = {}_3\phi_2\left[\left.\begin{matrix} Q^{-M}, abQ^{M+1}, Q^{-x} \\ aQ, Q^{-N} \end{matrix}\right| Q;Q\right], \qquad M \leqslant N. \tag{3.5}$$

Setting $a = Q^\alpha$, $b = Q^\beta$, we obtain the following form of the Q-Hahn polynomials:

$$\mathcal{Q}_M(x;\alpha,\beta,N;Q) = \sum_k \frac{(Q^{-M};Q)_k (Q^{\alpha+\beta+M+1};Q)_k (Q^{-x};Q)_k}{(Q^{\alpha+1};Q)_k (Q^{-N};Q)_k} \frac{Q^k}{(Q;Q)_k}. \tag{3.6}$$

Explicit formulae of the CGC are found in [23, 15]. Up to a multiplicative factor that is irrelevant to the present discussion, the CGC reads

$$\begin{aligned} C^{\ell_1\ell_2\ell}_{m_1m_2m} = N_1(\ell_1,\ell_2,\ell,m;q) &\sum_{m_1+m_2=m} (-1)^{(\ell_1-m_1)\lambda+\frac{1}{2}(\ell_1-m_1)(\ell_1-m_1+1)} \\ &\times q^{-\frac{1}{2}m_1(m+1)} \left(\frac{\{\ell_1-m_1\}_q!\{\ell_2-m_2\}_q!}{\{\ell_1+m_1\}_q!\{\ell_2+m_2\}_q!}\right)^{1/2} \\ &\times \sum_k (-1)^{k(\ell_1+\ell_2-m)+\frac{1}{2}k(k-1)} q^{\frac{1}{2}k(\ell+m+1)} \\ &\times \frac{\{\ell_1+\ell-m_2-k\}_q!\{\ell_2+m_2+k\}_q!}{\{\ell_1-\ell+m_2+k\}_q!\{\ell_2-m_2-k\}_q!\{\ell-m-k\}_q!\{k\}_q!}, \end{aligned} \tag{3.7}$$

where the index k runs over all non-negative integers maintaining the argument of $\{x\}_q$ non-negative. The part of summation over k can be regarded as a polynomial appearing in the CGC. To relate this to the Q-Hahn polynomial with $Q = -q$, we recast the factorials in (3.7) as shifted factorials:

$$\frac{\{A\}_q!}{\{A+k\}_q!} = q^{\frac{1}{4}k(2A+k-1)} \frac{(1+q)^k}{(Q^{A+1};Q)_k}, \tag{3.8}$$

$$\frac{\{A\}_q!}{\{A-k\}_q!} = (-1)^{\frac{1}{2}k(2A+3-k)} q^{\frac{1}{4}k(2A+3-k)} \frac{(Q^{-A};Q)_k}{(1+q)^k}, \tag{3.9}$$

where $Q = -q$. Apart from a multiplicative constant, the summation over k in (3.7) is now related to the polynomial form (3.6) with the assignment $Q = -q$:

$$\begin{aligned} \sum_k (\cdots) = &\frac{\{\ell_1+\ell-m_2\}_q!\{\ell_2+m_2\}_q!}{\{\ell_1-\ell+m_2\}_q!\{\ell_2-m_2\}_q!\{\ell-m\}_q!} \\ &\times \sum_k \frac{(Q^{-\ell_2+m_2};Q)_k (Q^{\ell_2+m_2+1};Q)_k (Q^{-\ell+m};Q)_k}{(Q^{\ell_1-\ell+m_2+1};Q)_k (Q^{-\ell_1-\ell+m_2};Q)_k} \frac{Q^k}{(Q;Q)_k}. \end{aligned} \tag{3.10}$$

The parameters of the Q-Hahn polynomial read

$$\begin{aligned} &\alpha = -\ell+\ell_1+m_2, \qquad \beta = \ell-\ell_1+m_2, \\ &N = \ell+\ell_1-m_2, \qquad x = \ell-m, \qquad M = \ell_2-m_2. \end{aligned} \tag{3.11}$$

3.2. Two even-dimensional representations

It is expected that in this case the decomposed irreducible spaces have odd dimensions. We consider representation in the tensored space $V_+^{(\ell_1)} \otimes V_-^{(\ell_2)}$ so that the constant η appearing in the eigenvalues of H is eliminated. The CGC for this case may be computed in the standard way as outlined below.

We start by determining the highest weight states in the direct product space $V_+^{(\ell_1)} \otimes V_-^{(\ell_2)}$. A highest weight state is a linear combination of the basis of $V_+^{(\ell_1)} \otimes V_-^{(\ell_2)}$:

$$e_\ell^\ell(\ell_1, \ell_2, \Lambda) = \sum_{m_1,m_2} C_{m_1,m_2} e_{m_1}^{\ell_1}(\lambda) \otimes e_{m_2}^{\ell_2}(\lambda). \tag{3.12}$$

The defining equations for the highest weight state read

$$\Delta(H) e_\ell^\ell(\ell_1, \ell_2, \Lambda) = \frac{\ell}{2} e_\ell^\ell(\ell_1, \ell_2, \Lambda), \qquad \Delta(V_+) e_\ell^\ell(\ell_1, \ell_2, \Lambda) = 0. \tag{3.13}$$

The first equality in (3.13) puts a constraint on the summation in (3.12). Implying the identity

$$\Delta(H) e_\ell^\ell(\ell_1, \ell_2, \Lambda) = \sum_{m_1,m_2} \frac{1}{2}(m_1 + m_2) C_{m_1,m_2} e_{m_1}^{\ell_1}(\lambda) \otimes e_{m_2}^{\ell_2}(\lambda) = \frac{\ell}{2} e_\ell^\ell(\ell_1, \ell_2, \Lambda),$$

it necessitates that the summation must obey the constraint $m_1 + m_2 = \ell$. As both m_1 and m_2 are half-integers, ℓ takes the integral values. The second equality in (3.13) produces a recurrence relation:

$$\begin{aligned} &q^{-\frac{1}{2}m_2} \sqrt{\{\ell_1 - m_1\}_q \{\ell_1 + m_1 + 1\}_q} C_{m_1,m_2} \\ &\qquad = (-1)^{\ell_1 - m_1 + \lambda + 1} q^{\frac{1}{2}(m_1+1)} \sqrt{\{\ell_2 + m_2\}_q \{\ell_2 - m_2 + 1\}_q} C_{m_1+1,m_2-1}. \end{aligned} \tag{3.14}$$

The recurrence relation may be solved explicitly:

$$\begin{aligned} C_{m_1,m_2} &= (-1)^{(\ell_1 - m_1)\lambda + \frac{1}{2}(\ell_1 - m_1)(\ell_1 - m_1 - 1)} q^{\frac{1}{2}(\ell+1)(\ell_1 - m_1)} \\ &\quad \times \left(\frac{\{\ell_1 + \ell_2 - \ell\}_q! \{\ell_1 + m_1\}_q! \{\ell_2 + m_2\}_q!}{\{2\ell_1\}_q! \{-\ell_1 + \ell_2 + \ell\}_q! \{\ell_1 - m_1\}_q! \{\ell_2 - m_2\}_q!} \right)^{1/2} C_{\ell_1, \ell - \ell_1}. \end{aligned} \tag{3.15}$$

The highest weight vector has been determined uniquely up to an overall factor $C_{\ell,\ell-\ell_1}$ that may be obtained by using the normalization. As is unnecessary for our purpose, we leave the constant undetermined.

Other states in $V_+^{(\ell_1)} \otimes V_-^{(\ell_2)}$ are obtained by repeated application of $\Delta(V_-)$ on the highest weight state:

$$\Delta(V_-)^{\ell - m} e_\ell^\ell(\ell_1, \ell_2, \Lambda) = \sum_{m_1 + m_2 = \ell} C_{m_1,m_2} \Delta(V_-)^{\ell - m} e_{m_1}^{\ell_1}(\lambda) \otimes e_{m_2}^{\ell_2}(\lambda). \tag{3.16}$$

The state (3.16) may be easily recognized as an eigenstate of the operator $\Delta(H)$ with the eigenvalue $m/2$. To express the state (3.16) as a linear combination of $e_{m_1}^{\ell_1}(\lambda) \otimes e_{m_2}^{\ell_2}(\lambda)$, we expand $\Delta(V_-)^{\ell - m}$ by using the binomial theorem for anti-commuting objects. For q-anti-commuting operators subject to $qAB + BA = 0$, the expansion reads

$$(A + B)^n = \sum_{k=0}^{n} q^{\frac{1}{2}k(n-k)} \frac{\{n\}_q!}{\{k\}_q! \{n-k\}_q!} A^k B^{n-k}. \tag{3.17}$$

Setting $A = q^H \otimes V_-$, $B = V_- \otimes q^{-H}$, we apply the expansion (3.17) in (3.16). Following a redefinition of the summation variables, we obtain the expression of CGC as

$$\begin{aligned} C_{m_1 m_2 m}^{\ell_1 \ell_2 \ell} &= N_2(\ell_1, \ell_2, \ell, m; q) \sum_{m_1 + m_2 = m} (-1)^{(\ell_1 - m_1)\lambda + \frac{1}{2}(\ell_1 - m_1)(\ell_1 - m_1 - 1)} \\ &\quad \times q^{-\frac{1}{2}m_1(m+1)} \left(\frac{\{\ell_1 - m_1\}_q! \{\ell_2 - m_2\}_q!}{\{\ell_1 + m_1\}_q! \{\ell_2 + m_2\}_q!} \right)^{1/2} \\ &\quad \times \sum_k (-1)^{k(\ell_1 + \ell_2 - m) + \frac{1}{2}k(k+1)} q^{\frac{1}{2}k(\ell + m + 1)} \\ &\quad \times \frac{\{\ell_1 + \ell - m_2 - k\}_q! \{\ell_2 + m_2 + k\}_q!}{\{\ell_1 - \ell + m_2 + k\}_q! \{\ell_2 - m_2 - k\}_q! \{\ell - m - k\}_q! \{k\}_q!}. \end{aligned} \tag{3.18}$$

This CGC is almost same as (3.7) discussed in the previous subsection, except for a sign factor that originates from the difference in the phases between the odd- and even-dimensional representations, given in (2.7) and (2.10), respectively. Moreover, the sign difference in the factors comprising the sum over the index k disappears when the expression (3.18) is recast in terms of the shifted factorials. This leads to identical sums in (3.10) and (3.18) on the index k. We may, therefore, immediately conclude that the CGC in (3.18) are related to the Q-Hahn polynomials with $Q = -q$, and the values of the parameters are given by (3.11).

The above construction of the eigenstates of $\Delta(H)$ is just the standard procedure of highest weight construction leading to a multiplet of $2\ell + 1$ states from $e^{\ell}_{\ell}(\ell_1, \ell_2, \Lambda)$ by repeated actions of $\Delta(V_-)$. The $2\ell + 1$ states are linearly independent, since they are the eigenvectors of $\Delta(H)$ with different eigenvalues. Therefore, they form a basis of an invariant subspace in $V_+^{(\ell_1)} \otimes V_-^{(\ell_2)}$. It is an easy task to verify that eigenstates (3.16) belonging to different values of ℓ are linearly independent. It also follows that, as $\ell \geqslant 0$, its possible values are $\ell_1 + \ell_2, \ell_1 + \ell_2 - 1, \ldots, |\ell_1 - \ell_2|$. The total number of the eigenstates of $\Delta(H)$, of course, coincides with the dimension of $V_+^{(\ell_1)} \otimes V_-^{(\ell_2)}$:

$$\sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} (2\ell + 1) = (2\ell_1 + 1)(2\ell_2 + 1).$$

Therefore, all eigenstates of $\Delta(H)$ form a basis of $V_+^{(\ell_1)} \otimes V_-^{(\ell_2)}$. In the present case, we have thus proved the decomposition (3.1).

3.3. Odd- and even-dimensional representations

We consider representations in the space $V^{(\ell_1)} \otimes V_{\pm}^{(\ell_2)}$, where the first (second) space in the tensor product is odd-(even-)dimensional. The CGC for this case can be computed in the same way as in the previous subsection. We here list some corresponding formulae and omit the computational details. A highest weight state in $V^{(\ell_1)} \otimes V_{\pm}^{(\ell_2)}$ has the form of (3.12) where the summation variables run under the constraint $m_1 + m_2 = \ell$. Since m_1 (m_2) is an integer (half-integer), ℓ takes a half-integral value and the constant η remains in the expression of the weight:

$$\Delta(H)e^{\ell}_{\ell}(\ell_1, \ell_2, \Lambda) = \tfrac{1}{2}(\ell \pm \eta)e^{\ell}_{\ell}(\ell_1, \ell_2, \Lambda). \tag{3.19}$$

The highest weight condition determines the coefficient C_{m_1,m_2}:

$$\begin{aligned} C_{m_1,m_2} &= (-1)^{(\ell_1-m_1)\lambda+\frac{1}{2}(\ell_1-m_1)(\ell_1-m_1-1)} q^{\frac{1}{2}(\ell+1\pm\eta)(\ell_1-m_1)} \\ &\quad \times \left(\frac{1}{\{2\ell_1\}_q!} \frac{\{\ell_1+\ell_2-\ell\}_q!\{\ell_1+m_1\}_q!\{\ell_2+m_2\}_q!}{\{-\ell_1+\ell_2+\ell\}_q!\{\ell_1-m_1\}_q!\{\ell_2-m_2\}_q!} \right)^{1/2} C_{\ell_1,\ell-\ell_1}. \end{aligned} \tag{3.20}$$

The factor $C_{\ell_1,\ell-\ell_1}$ may be determined by normalization of the highest weight states. Other states in $V^{(\ell_1)} \otimes V_{\pm}^{(\ell_2)}$ are obtained by repeated applications of $\Delta(V_-)$ on the highest weight states. The CGC for this case may be read off from the expression of state vectors:

$$\begin{aligned} C^{\ell_1\ell_2\ell}_{m_1m_2m} &= N_3(\ell_1, \ell_2, \ell, m; q) \sum_{m_1+m_2=m} (-1)^{(\ell_1-m_1)\lambda+\frac{1}{2}(\ell_1-m_1)(\ell_1-m_1-1)} \\ &\quad \times q^{-\frac{1}{2}m_1(m+1\pm\eta)} \left(\frac{\{\ell_1-m_1\}_q!\{\ell_2-m_2\}_q!}{\{\ell_1+m_1\}_q!\{\ell_2+m_2\}_q!} \right)^{1/2} \\ &\quad \times \sum_k (-1)^{k(\ell_1+\ell_2-m)+\frac{1}{2}k(k-1)} q^{\frac{1}{2}k(\ell+m+1)} \\ &\quad \times \frac{\{\ell_1+\ell-m_2-k\}_q!\{\ell_2+m_2+k\}_q!}{\{\ell_1-\ell+m_2+k\}_q!\{\ell_2-m_2-k\}_q!\{\ell-m-k\}_q!\{k\}_q!}. \end{aligned} \tag{3.21}$$

The factor that includes the sum over k in (3.21) may be converted into identical form as (3.10). Thus, the polynomial part of the CGC in (3.21) leads to the Q-Hahn polynomial with $Q = -q$ and the parameters listed in (3.11). The same discussion as in the previous subsection completes the proof of the decomposition (3.1).

4. Even-dimensional representations of $OSp_q(1/2)$ and little Q-Jacobi polynomials

In this section, we compute the even-dimensional representations of the quantum supergroup $\mathcal{A} \equiv OSp_q(1/2)$ by choosing a different basis set, and adopting a different method from [12]. We remark that precise theory of matrix representations of quantum group has been developed in [25], and that the odd-dimensional representations of $\mathcal{A}$ have been obtained in [12, 13]. In [12], an algebra $\mathcal{A}(\sigma)$ generated by 2×2 quantum supermatrix is set in the beginning, and later its dual algebra is identified with $\mathcal{U}$. The representations of the algebra $\mathcal{A}(\sigma)$ are obtained in a way parallel to [4]. In contrast to this approach, we start with the algebra $\mathcal{U}$, and then determine its dual basis. It follows the construction of the universal $\mathcal{T}$-matrix, and the representations of $\mathcal{A}$ are readily obtained by taking matrix elements of the universal $\mathcal{T}$-matrix in the representation space of the algebra $\mathcal{U}$. In addition to the easy and clear mechanism of our construction, the use of the universal $\mathcal{T}$-matrix imparts the following advantages: (i) the algebraic structure of $\mathcal{A}$ is made transparent in the construction of the universal $\mathcal{T}$-matrix, as its basis set is determined explicitly; (ii) the nontrivial contribution of parity odd elements of $\mathcal{A}$ to representations can be easily read off from the form of the universal $\mathcal{T}$-matrix, that is, distinction from Lie supergroup $OSp(1/2)$ is emphasized.

We divide this section in two parts. The first part contains a summary of the basis of the algebra $\mathcal{A}$, and we also quote the universal $\mathcal{T}$-matrix from [13], where its detailed construction employing the Hopf duality between the $\mathcal{U}$ and $\mathcal{A}$ algebras is given. The second part is devoted to the computation of the even-dimensional representations of $\mathcal{A}$ and their relation to the little Q-Jacobi polynomials.

4.1. $OSp_q(1/2)$ and universal $\mathcal{T}$-matrix

The algebra $\mathcal{A}$, introduced in [19, 20], is a Hopf algebra dual to the algebra $\mathcal{U}$. Two Hopf algebras $\mathcal{U}$ and $\mathcal{A}$ are in duality if there exists a doubly nondegenerate bilinear form $\langle , \rangle : \mathcal{A} \otimes \mathcal{U} \to \mathbb{C}$ such that, for $(\mathsf{a}, \mathsf{b}) \in \mathcal{A}$, $(\mathsf{u}, \mathsf{v}) \in \mathcal{U}$,

$$\begin{aligned} &\langle \mathsf{a}, \mathsf{uv} \rangle = \langle \Delta_{\mathcal{A}}(\mathsf{a}), \mathsf{u} \otimes \mathsf{v} \rangle, \qquad \langle \mathsf{ab}, \mathsf{u} \rangle = \langle \mathsf{a} \otimes \mathsf{b}, \Delta_{\mathcal{U}}(\mathsf{u}) \rangle, \\ &\langle \mathsf{a}, 1_{\mathcal{U}} \rangle = \epsilon_{\mathcal{A}}(\mathsf{a}), \qquad \langle 1_{\mathcal{A}}, \mathsf{u} \rangle = \epsilon_{\mathcal{U}}(\mathsf{u}), \qquad \langle \mathsf{a}, S_{\mathcal{U}}(\mathsf{u}) \rangle = \langle S_{\mathcal{A}}(\mathsf{a}), \mathsf{u} \rangle. \end{aligned} \tag{4.1}$$

The algebra $\mathcal{A}$ is generated by three elements, which are dual to the generators of the algebra $\mathcal{U}$:

$$\langle x, V_+ \rangle = 1, \qquad \langle z, H \rangle = 1, \qquad \langle y, V_- \rangle = 1. \tag{4.2}$$

Thus, x and y are of odd parity, while z is even. The generating elements satisfy the commutation relations:

$$\{x, y\} = 0, \qquad [z, x] = 2 \ln q x, \qquad [z, y] = 2 \ln q y. \tag{4.3}$$

Let the ordered monomials $E_{k\ell m} = V_+^k H^\ell V_-^m$, $(k, \ell, m) \in (0, 1, 2, \ldots)$ be the basis elements of the algebra $\mathcal{U}$. The basis elements $e^{k\ell m}$ of the dual Hopf algebra $\mathcal{A}$ follow the relation

$$\langle e^{k\ell m}, E_{k'\ell'm'} \rangle = \delta^k_{k'} \delta^\ell_{\ell'} \delta^m_{m'}. \tag{4.4}$$

The generating elements of the algebra $\mathcal{A}$ may be identified as $x = e^{100}$, $y = e^{001}$ and $z = e^{010}$. The basis elements $e^{k\ell m}$ are ordered polynomials in the generating elements:

$$e^{k\ell m} = \frac{x^k}{\{k\}_q!}\frac{(z+(k-m)\ln q)^\ell}{\ell!}\frac{y^m}{\{m\}_{q^{-1}}!}. \tag{4.5}$$

Using the duality structure, full Hopf structure of the algebra $\mathcal{A}$ has been obtained in [13]. We, however, do not list them here as it is not used in the subsequent discussions.

The notion of the universal $\mathcal{T}$-matrix is a key feature capping the Hopf duality structure. The universal $\mathcal{T}$-matrix for the superalgebra is defined by

$$\mathcal{T}_{e,E} = \sum_{k\ell m}(-1)^{\widehat{e^{k\ell m}}(\widehat{e^{k\ell m}}-1)/2} e^{k\ell m}\otimes E_{k\ell m}, \tag{4.6}$$

where the parity of basis elements is same for the two Hopf algebras $\mathcal{U}$ and $\mathcal{A}$:

$$\widehat{e^{k\ell m}} = \widehat{E_{k\ell m}} = k+m. \tag{4.7}$$

Consequently, the duality relations (4.1) may be concisely expressed [26] in terms of the $\mathcal{T}$-matrix as

$$\begin{aligned} &\mathcal{T}_{e,E}\mathcal{T}_{e',E} = \mathcal{T}_{\Delta(e),E}, \qquad && \mathcal{T}_{e,E}\mathcal{T}_{e,E'} = \mathcal{T}_{e,\Delta(E)},\\ &\mathcal{T}_{\epsilon(e),E} = \mathcal{T}_{e,\epsilon(E)} = 1, && \mathcal{T}_{S(e),E} = \mathcal{T}_{e,S(E)}, \end{aligned} \tag{4.8}$$

where e and e' (E and E') refer to the two identical copies of algebra $\mathcal{A}$ ($\mathcal{U}$).

Our explicit listing of the complete set of dual basis elements in (4.5) allows us to obtain the universal $\mathcal{T}$-matrix as an operator-valued function in a closed form:

$$\begin{aligned} \mathcal{T}_{e,E} &= \left(\sum_{k=0}^{\infty}\frac{(x\otimes V_+q^H)^k}{\left[k\right]_q!}\right)\exp(z\otimes H)\left(\sum_{m=0}^{\infty}\frac{(y\otimes q^{-H}V_-)^m}{\left[m\right]_{q^{-1}}!}\right)\\ &\equiv {}^{\times}_{\times}\mathcal{E}\,\mathrm{xp}_q(x\otimes V_+q^H)\exp(z\otimes H)\mathcal{E}\,\mathrm{xp}_{q^{-1}}(y\otimes q^{-H}V_-)^{\times}_{\times}, \end{aligned} \tag{4.9}$$

where we have introduced a deformed exponential that is characteristic of the quantum $OSp_q(1/2)$ supergroup:

$$\mathcal{E}\,\mathrm{xp}_q(\mathcal{X}) \equiv \sum_{n=0}^{\infty}\frac{\mathcal{X}^n}{\left[n\right]_q!}, \qquad \left[n\right]_q = \frac{1-(-1)^nq^n}{1+q}. \tag{4.10}$$

The operator ordering has been explicitly indicated in (4.9). In [27], using the Gauss decomposition of the fundamental representation, a universal $\mathcal{T}$-matrix for $\mathcal{U}$ is given in terms of the standard q-exponential instead of the deformed exponential (4.10) characterizing quantum supergroup. In the classical $q\to 1$ limit, the universal $\mathcal{T}$-matrix (4.9) yields [13] the group element of the undeformed supergroup $OSp(1/2)$. As the nilpotency relations $x^2 = 0$, $y^2 = 0$ hold in the classical regime, we assume the following finite limits:

$$\lim_{q\to 1}\frac{x^2}{q-1} = \mathfrak{x}, \qquad \lim_{q\to 1}\frac{y^2}{q^{-1}-1} = \mathfrak{y}. \tag{4.11}$$

It then follows that in this limit the universal $\mathcal{T}$-matrix (4.9) reduces to an element of the classical supergroup $OSp(1/2)$:

$$\mathcal{G} = (1\otimes 1 + x\otimes V_+)\exp\left(\mathfrak{x}\otimes V_+^2\right)\exp(z\otimes H)\exp(\mathfrak{y}\otimes V_-^2)(1\otimes 1 + y\otimes V_-). \tag{4.12}$$

The well-known existence of the classical $SL(2)$ subgroup structure generated by the elements $\left(V_\pm^2, H\right)$ of the undeformed $osp(1/2)$ algebra is evident from (4.12). In fact, the correct limiting structure (4.12) emphasizes that the quantum universal $\mathcal{T}$-matrix embodies the duality between

the $\mathcal{U}$ and $\mathcal{A}$ algebras in a way that runs parallel to the familiar dual kinship between the classical Lie algebras and groups. Mappings from the universal $\mathcal{T}$-matrix to the universal $\mathcal{R}$-matrix of $\mathcal{U}$ also exist [13].

4.2. Representation matrices

The closed form of the universal $\mathcal{T}$-matrix in (4.9) can be used to compute the representation matrices of the quantum supergroup $\mathcal{A}$ as has been done in [13]. To be explicit, we construct the representations of $\mathcal{A}$ by evaluating the matrix elements of the universal $\mathcal{T}$-matrix on $V^{(\ell)}$ defined in section 2:

$$\begin{aligned} T^{\ell}_{m'm}(\lambda) &= \left(e^{\ell}_{m'}(\lambda), \mathcal{T}_{e,E} e^{\ell}_{m}(\lambda)\right) \\ &= \sum_{abc} (-1)^{(a+c)(a+c-1)/2+(a+c)(\ell-m'+\lambda)} e^{abc} \left(e^{\ell}_{m'}(\lambda), E_{abc} e^{\ell}_{m}(\lambda)\right). \end{aligned} \tag{4.13}$$

Assuming the completeness of the basis vectors $e^{\ell}_{m}(\lambda)$, we may verify the properties

$$\Delta(T^{\ell}_{m'm}(\lambda)) = \sum_{k} T^{\ell}_{m'k}(\lambda) \otimes T^{\ell}_{km}(\lambda), \qquad \epsilon\left(T^{\ell}_{m'm}(\lambda)\right) = \delta_{m'm}, \tag{4.14}$$

which imply that the matrix elements (4.13) satisfy the axiom of comodule [1]. We may, therefore, regard $T^{\ell}_{m'm}(\lambda)$ as the $(2\ell+1)$-dimensional matrix representation of the algebra $\mathcal{A}$. We proceed to compute the matrix elements for the even-dimensional representations (2.10) in the present section. The explicit listing of the basis elements of $\mathcal{A}$ in (4.5) renders the computation of the matrix elements straightforward. The computation is carried out by using two identities obtained by the repeated use of (2.10):

$$V^{a}_{+} e^{\ell}_{m}(\lambda) = (\pm 1)^{a} \left(\frac{1}{\{2\}^{a}_{q}} \frac{\{\ell-m\}_q!}{\{\ell+m\}_q!} \frac{\{\ell+m+a\}_q!}{\{\ell-m-a\}_q!} \right)^{1/2} e^{\ell}_{m+a}(\lambda) \tag{4.15}$$

and

$$V^{c}_{-} e^{\ell}_{m}(\lambda) = \mathrm{i}^{c} (-1)^{c(\ell-m)+c(c-1)/2} \left(\frac{1}{\{2\}^{c}_{q}} \frac{\{\ell+m\}_q!}{\{\ell-m\}_q!} \frac{\{\ell-m+c\}_q!}{\{\ell+m-c\}_q!} \right)^{1/2} e^{\ell}_{m-c}(\lambda). \tag{4.16}$$

We just quote the final result as

$$\begin{aligned} T^{\ell}_{m'm} = {} & (\pm 1)^{m'-m} (-1)^{\frac{1}{2}(m'-m)(m'-m-1)+(m'-m)(\ell-m'+\lambda)} \, \mathrm{e}^{\pm \frac{\pi}{4}(m'-m)\mathrm{i}} \\ & \times q^{\frac{1}{2}m(m'-m)} \left(\frac{1}{\{2\}^{m'-m}_{q}} \frac{\{\ell+m\}_q! \{\ell+m'\}_q!}{\{\ell-m\}_q! \{\ell-m'\}_q!} \right)^{1/2} \\ & \times \sum_{c} (\pm \mathrm{i})^{c} (-1)^{c(\ell-m-1)} \frac{q^{-\frac{1}{2}c(m'-m)}}{\{2\}^{c}_{q}} \frac{\{\ell-m+c\}_q!}{\{\ell+m-c\}_q!} \\ & \times \frac{x^{m'-m+c}}{\{m'-m+c\}_q!} \exp\left(\frac{m-c}{2} z \pm \frac{\pi \mathrm{i}}{4 \ln q} z \right) \frac{y^{c}}{\{c\}_q!}, \end{aligned} \tag{4.17}$$

where the index c runs over all non-negative integers maintaining the argument of $\{x\}_q$ non-negative.

We now turn our attention to the polynomial structure built into the general matrix element (4.17) in terms of the variable

$$\zeta = -\frac{q^{-1/2}}{\{2\}_q} x \, \mathrm{e}^{-z/2} y. \tag{4.18}$$

We note that the variable (4.18) differs in sign from the corresponding one for the odd-dimensional case. To demonstrate this, the product of generators in (4.17) for the case $m' - m \geqslant 0$ may be rearranged as follows:

$$x^{m'-m+c} \exp\left(\frac{m-c}{2}z \pm \frac{\pi \mathrm{i}}{4\ln q}z\right) y^c = (\mp \mathrm{i})^c q^{-mc} x^{m'-m} \exp\left(\frac{m}{2}z \pm \frac{\pi \mathrm{i}}{4\ln q}z\right) x^c \,\mathrm{e}^{-cz/2} y^c.$$

The matrix element $T^{\ell}_{m'm}(\lambda)$ may now be succinctly expressed as a polynomial structure:

$$\begin{aligned} T^{\ell}_{m'm} = {} & (\pm 1)^{m'-m}(-1)^{(m'-m)(m'-m-1)/2+(m'-m)(\ell-m'+\lambda)} \\ & \times \mathrm{e}^{\pm\frac{\pi}{4}(m'-m)\mathrm{i}} \frac{q^{\frac{1}{2}m(m'-m)}}{\{m'-m\}_q!} \left(\frac{1}{\{2\}_q^{m'-m}} \frac{\{\ell-m\}_q!\{\ell+m'\}_q!}{\{\ell+m\}_q!\{\ell-m'\}_q!}\right)^{1/2} \\ & \times x^{m'-m} \exp\left(\frac{m}{2}z \pm \frac{\pi \mathrm{i}}{4\ln q}z\right) P^{\ell}_{m'm}(\zeta). \end{aligned} \tag{4.19}$$

The polynomial $P^{\ell}_{m'm}(\zeta)$ in the variable ζ is defined by

$$\begin{aligned} P^{\ell}_{m'm}(\zeta) = {} & \sum_c (-1)^{c(\ell-m)+\frac{1}{2}c(c-1)} q^{-\frac{1}{2}c(m'+m-1)} \\ & \times \frac{\{m'-m\}_q!\{l+m\}_q!\{\ell-m+c\}_q!}{\{m'-m+c\}_q!\{\ell+m-c\}_q!\{\ell-m\}_q!\{c\}_q!} \zeta^c, \end{aligned} \tag{4.20}$$

where the index c runs over all non-negative integers maintaining the arguments of $\{x\}_q$ non-negative. The polynomial (4.20) is identical to the one appearing in the odd-dimensional case [13]. For the case $m' - m \leqslant 0$, we make a replacement of the summation index c with $a = m' - m + c$. Rearrangement of the generators now provides the following expression of the general matrix element:

$$\begin{aligned} T^{\ell}_{m'm} = {} & \mathrm{i}^{m-m'}(-1)^{\frac{1}{2}(m-m')(m-m'+1)+(m-m')\lambda} \\ & \times \mathrm{e}^{\mp\frac{\pi}{4}(m-m')\mathrm{i}} \frac{q^{-\frac{1}{2}m'(m-m')}}{\{m-m'\}_q!} \left(\frac{1}{\{2\}_q^{m-m'}} \frac{\{\ell+m\}_q\{\ell-m'\}_q}{\{\ell-m\}_q\{\ell+m'\}_q}\right)^{1/2} \\ & \times \exp\left(\frac{m'}{2}z \pm \frac{\pi \mathrm{i}}{4\ln q}z\right) y^{m-m'} P^{\ell}_{m'm}(\zeta), \end{aligned} \tag{4.21}$$

where the polynomial $P^{\ell}_{m'm}(\zeta)$ for $m' - m \leqslant 0$ is defined by

$$\begin{aligned} P^{\ell}_{m'm}(\zeta) = {} & \sum_a (-1)^{a(\ell-m')+\frac{1}{2}a(a-1)} q^{-\frac{1}{2}a(m'+m-1)} \\ & \times \frac{\{m-m'\}_q!\{l+m'\}_q!\{\ell-m'+a\}_q!}{\{m-m'+a\}_q!\{\ell+m'-a\}_q!\{\ell-m'\}_q!\{a\}_q!} \zeta^a. \end{aligned} \tag{4.22}$$

The polynomial (4.22) is also identical to the corresponding one obtained in [13].

As seen above, the even-dimensional representations of the algebra $\mathcal{A}$ have the same polynomial structure as the odd-dimensional ones, though, between these two cases, the variable ζ differs by a sign. Thus, the polynomials (4.20) and (4.22) are identified to the little Q-Jacobi polynomials with $Q = -q$. The little Q-Jacobi polynomials are defined via ${}_2\phi_1$ [24]:

$$\begin{aligned} p_m^{(\alpha,\beta)}(z) &= {}_2\phi_1(Q^{-m}, Q^{\alpha+\beta+m+1}; Q^{\alpha+1}; Q; Qz) \\ &= \sum_n \frac{(Q^{-m}; Q)_n (Q^{\alpha+\beta+m+1}; Q)_n}{(Q^{\alpha+1}; Q)_n (Q; Q)_n} (Qz)^n. \end{aligned} \tag{4.23}$$

Rewriting our polynomials (4.20) and (4.22) in terms of the shifted factorial with $Q = -q$, their identification is readily obtained. For the choice $m' - m \geqslant 0$, the polynomial structure reads

$$P^{\ell}_{m'm}(\zeta) = \sum_a \frac{(Q^{-\ell-m};Q)_a(Q^{\ell-m+1};Q)_a}{(Q^{m'-m+1};Q)_a(Q;Q)_a}(Q\zeta)^a = p^{(m'-m,-m'-m)}_{\ell+m}(\zeta), \tag{4.24}$$

and for the $m' - m \leqslant 0$ case its identification is given by

$$P^{\ell}_{m'm}(\zeta) = \sum_a \frac{(Q^{-\ell-m'};Q)_a(Q^{\ell-m'+1};Q)_a}{(Q^{m-m'+1};Q)_a(Q;Q)_a}(Q\zeta)^a = p^{(m-m',-m'-m)}_{\ell+m'}(\zeta). \tag{4.25}$$

The even- and odd-dimensional representations of the algebra $\mathcal{A}$ have almost the same form. The fundamental difference of these is the factor $\exp(\pm\eta z/2)$ appearing in the even-dimensional representations. The factor is not well defined in the classical limit of $q \to 1$. This feature of the even-dimensional representations of the algebra $\mathcal{A}$ owes its genesis from the corresponding one of its dual algebra $\mathcal{U}$.

To connect the results in this section to that of [12], we consider the representation specified by $\ell = \frac{1}{2}$, $\lambda = 0$. We denote the matrix elements as follows:

$$\begin{aligned} a &= T^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} = \exp\left(\frac{z}{4} \pm \frac{\eta z}{2}\right)(1+\zeta), & b &= T^{\frac{1}{2}}_{\frac{1}{2}-\frac{1}{2}} = \pm\frac{\mathrm{e}^{\mp\frac{\pi}{4}\mathrm{i}}}{q^{\frac{1}{4}}\{2\}^{\frac{1}{2}}_q}xd, \\ c &= T^{\frac{1}{2}}_{-\frac{1}{2}\frac{1}{2}} = \mathrm{i}\frac{\mathrm{e}^{\mp\frac{\pi}{4}\mathrm{i}}}{q^{\frac{1}{4}}\{2\}^{\frac{1}{2}}_q}\mathrm{d}y, & d &= T^{\frac{1}{2}}_{-\frac{1}{2}-\frac{1}{2}} = \exp\left(-\frac{z}{4} \pm \frac{\eta z}{2}\right). \end{aligned} \tag{4.26}$$

The commutation relations satisfied by the matrix elements may be immediately derived:

$$\begin{aligned} ab &= \pm\mathrm{i}q^{\frac{1}{2}}ba, & ac &= \pm\mathrm{i}q^{\frac{1}{2}}ca, & bc &= -cb, \\ bd &= \mp\mathrm{i}q^{\frac{1}{2}}db, & cd &= \mp\mathrm{i}q^{\frac{1}{2}}dc, & [a,d] &= -(1+q)bc. \end{aligned} \tag{4.27}$$

The central element $ad + qbc$ commutes with a, d and anti-commutes with b, c. Thus, the representations specified by $\ell = \frac{1}{2}$, $\lambda = 0$ are precisely same as the algebra $\mathcal{A}(\sigma)$ used in [12].

5. Even-dimensional covariant spaces of $OSp_q(1/2)$

It has been observed [15, 29] that noncommutative spaces covariant under the action of a finite-dimensional representation of quantum groups such as $SL_q(2)$ or $OSp_q(1/2)$ may be obtained by using the CGC. The method developed in [15, 29] is outlined below. We introduce an algebraic structure on a given representation space $V^{(\ell)}$. Namely, assuming a multiplication map $\mu : V^{(\ell)} \otimes V^{(\ell)} \to V^{(\ell)}$, we determine a consistent set of commutation relations among bases of $V^{(\ell)}$ that may be regarded as the generators of noncommutative spaces. Specifically, for the highest weight ℓ representation of $OSp_q(1/2)$, we construct the following composite object:

$$E^L_M(\Lambda) = \sum_{\substack{m_1,m_2 \\ m_1+m_2=M}} C^{\ell\ \ell L}_{m_1 m_2 M} e^{\ell}_{m_1}(\lambda) e^{\ell}_{m_2}(\lambda), \tag{5.1}$$

where $\Lambda = 2\ell - L \pmod 2$, where ℓ is of integral or half-integral value. Then it may be proved that the following relations are covariant under the right coaction of the highest weight

ℓ representation of $OSp_q(1/2)$:

$$E_0^0(0) = r, \tag{5.2}$$

$$E_M^\ell(\Lambda) = \xi e_M^\ell(\lambda), \tag{5.3}$$

$$E_M^L(\Lambda) = 0, \qquad (L \neq \ell, 0), \tag{5.4}$$

where r and ξ are parameters. In the $q \to 1$ limit $\xi \to 0$, and ξ is regarded as a Grassmann number if the parity of the two sets of vectors in (5.3) differ: $\Lambda \neq \lambda \pmod 2$.

Although we have obtained a set of covariant commutation properties, the simultaneous use of all relations from (5.2) to (5.4) gives an inconsistent result, since some of them do not have correct classical limits. In order to obtain a consistent covariant algebra, we have to make a choice regarding the relations to be used for defining the algebra. Then their consistency has to be verified. The consistency requirements are as follows:

(a) The constant r commutes with all generators.
(b) The associativity of products of generators need to be maintained.

Employing the above procedure in conjunction with the CGC given in (3.2), we now construct covariant noncommutative spaces of dimensions two and four.

Case 1. $\ell = \frac{1}{2}$.

The allowed values of L are 0 and 1. We rewrite the basis of $V^{(1/2)}$ as follows:

$$e_{1/2}^{1/2}(\lambda) \to x, \qquad e_{-1/2}^{1/2}(\lambda) \to y.$$

We first consider the case of $\lambda = 0$, when $x(y)$ is of even (odd) parity. Covariant relations for $L = 1$ obtained from (5.4) read

$$x^2 = y^2 = 0, \qquad q^{1/4}xy + q^{-1/4}yx = 0. \tag{5.5}$$

These relations are unacceptable as a definition of the covariant noncommutative space as there the even element x becomes nilpotent. We rather regard $L = 0$ relation obtained from (5.2) as a definition of the covariant space:

$$xy + q^{1/2}yx = r. \tag{5.6}$$

We now illustrate $\lambda = 1$ case where $x(y)$ is of odd (even) parity. One can see that $L = 1$ relations are rejected again by the same reason. We thus obtain a covariant space from (5.2):

$$xy - q^{1/2}yx = r. \tag{5.7}$$

Setting $r = 0$ in (5.6) or (5.7), the quantum superspaces found in the literatures (e.g. [30, 31]) are recovered. However, (5.6) or (5.7) gives the most general two-dimensional covariant superspaces.

Case 2. $\ell = \frac{3}{2}$.

The index L ranges the integral values from 3 to 0. We rewrite the basis of $V^{(3/2)}$ as

$$e_{3/2}^{3/2}(\lambda) \to x, \qquad e_{1/2}^{3/2}(\lambda) \to y, \qquad e_{-1/2}^{3/2}(\lambda) \to z, \qquad e_{-3/2}^{3/2}(\lambda) \to w.$$

We study the case of $\lambda = 0$, since the example $\lambda = 1$ yields almost identical results except for some sign differences. For the choice $\lambda = 0$, the generating elements $x, z(y, w)$ are of even (odd) parity. The results corresponding to $L = 3$ obtained from (5.4) contain an unacceptable

relation: $x^2 = 0$. We, therefore, discard these equations. The equations for the $L = 1, 2$ cases obtained from (5.4) provide six commutation relations and two additional constraints:

$$\begin{aligned} &xy + q^{3/2}yx = 0, \qquad xz - q^3zx = 0, \\ &(q^{-2} - 1 + q)xw - \{2\}_q wx + q^{-1/2}(q^2 - 1 + q^{-2})yz = 0, \\ &(q^2 - 1 + q^{-1})yz + q\{2\}_q zy + q^{1/2}\{3\}_q xw = 0, \\ &yw - q^3wy = 0, \qquad zw + q^{3/2}wz = 0 \end{aligned} \tag{5.8}$$

and

$$y^2 = q^{-3/2}\sqrt{\{3\}_q}xz, \qquad z^2 = q^{-3/2}\sqrt{\{3\}_q}yw. \tag{5.9}$$

It turns out that these relations do not satisfy the consistency condition (b). For instance, two ways of reversing xyz to zyx do not give identical result. We thus incorporate the relation corresponding to $L = 0$. We regard this relation as an additional constraint after setting $r = 0$ in (5.2), and thereby make the four-dimensional covariant space well defined. The constraint reads $yz = -q^{-3/2}\{3\}_q xw$. Employing this constraint we may simplify the commutation properties (5.8) and (5.9). The requirement that the simplified relations obey the consistency condition (b) is also verified. We, therefore, introduce the four-dimensional covariant space defined by the six commutation relations

$$\begin{aligned} &xy = -q^{3/2}yx, \qquad xz = q^3zx, \qquad xw = -q^{9/2}wx, \\ &yz = -q^{3/2}zy, \qquad yw = q^3wy, \qquad zw = -q^{3/2}wz, \end{aligned} \tag{5.10}$$

and three constraints

$$y^2 = q^{-3/2}\sqrt{\{3\}_q}xz, \qquad z^2 = q^{-3/2}\sqrt{\{3\}_q}yw, \qquad yz = -q^{-3/2}\{3\}_q xw. \tag{5.11}$$

6. Concluding remarks

We have seen intimate relations between the representations of the algebras $\mathcal{U}, \mathcal{A}$ and basic hypergeometric functions. Existence of even-dimensional representations makes the representation theory of the quantized $osp(1/2)$ algebra richer and more interesting than the one of the classical Lie superalgebra $osp(1/2)$. Especially, the fundamental representation of $osp(1/2)$ is three-dimensional, while the corresponding representation of $\mathcal{U}$ can be further decomposed into the product of two-dimensional ones. In other words, quasi-particles described by the three-dimensional representation of $\mathcal{U}$ can be regarded as a composite of more fundamental objects. Such situation, hopefully, may be realized in some physical models. A byproduct of the even-dimensional representations of the $osp_q(1/2)$ algebra is that new noncommutative spaces covariant under the coaction of the quantum group $OSp_q(1/2)$ may be constructed via the Clebsch–Gordan decomposition. The representations of these noncommutative spaces for the root of unity values of q, for instance, may be relevant for some physical problems.

Turning to the representations of Lie superalgebras, little seems to be known about their relations to hypergeometric functions. This may be explained by the appearance of the $Q = -q$ polynomials for the case of the algebras $\mathcal{U}$ and $\mathcal{A}$. The classical limit of such polynomials has somewhat complicated structure as they have to be evaluated at $Q = -1$. It may be difficult to find such polynomials starting from the representations of classical objects such as $osp(1/2)$ and $OSp(1/2)$. In this sense, the study of the representations of the quantum superalgebras gives deeper understanding of the representation theory of the Lie superalgebras. It is known that there is a one-to-one correspondence between the finite-dimensional representations of $osp(1/2n)$ and $so(2n + 1)$ except for the spinorial ones. For

quantum algebras, this is explained [14] by the isomorphism between $U_q[osp(1/2n)]$ and $U_{-q}[so(2n+1)]$, which holds on the non-spinorial representation spaces. Our work confirms that for the even-dimensional representations for which the said isomorphism is not known are still characterized by the $Q = -q$ polynomials. This may be a more general feature of the quantum supergroups.

Acknowledgments

We thank S Blumen for introducing us to the work of Zou [21]. We also thank the referee for useful comments. The work of NA is partially supported by the grants-in-aid from JSPS, Japan (contract no 15540132). Another author (SSNM) is supported by the University Grants Commission, Government of India.

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